Functions of Bounded Variation

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Content

1 Introduction

Since last semester, I have studied *Advanced Calculus* in Department of Mathematics, Tsing Hua University. In the course, *Principles of Mathematical Analysis* [2] was assigned to be our textbook and we learned the definition of the Riemann-Stieltjes integral in the book. However, the book only introduces the Riemann-Stieltjes integral of a function f with respect to a monotonically increasing function α. After some discussions with my advisor, I discovered that there is a more general definition of Riemann-Stieltjes integral, which does not have any requirements on α. Moreover, I found that the text *Mathematical Analysis* [1] written by Apostol, T.M. contains more details concerning the case when α is of bounded variation. This gives me the motivation to explore the properties of functions of bounded variation. In this article, it is separated into two parts, first of which contains some vital ideas of bounded variation functions. The remaining part includes some problems provided in the text *Mathematical Analysis*.

2 Propositions of Bounded Variation Functions

In the beginning of this section, we will introduce some concepts of functions of bounded variation. For brevity and clarity, we should introduce some notations and terminologies to avoid misunderstanding. We confine our attention to real-valued functions defined on bounded interval like $[a, b]$. Unless otherwise stated, $f, g, h, \dots$ would stand for such functions mentioned above.

Definition 1 (partitions and refinements)**.** Let $[a, b]$ be a given bounded interval. A set of finite points

$$P = \{x_0, x_1, \dots, x_n\},$$

which satisfies

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,$$

is called a partition of $[a, b]$. We usually write

$$\Delta x_i = x_i - x_{i-1} \quad (i = 1, \dots, n).$$

A partition P^* is called a refinement of P if $P^* \supset P$.

Definition 2 (functions of bounded variation)**.** Let f be a real-valued function defined on $[a, b]$, and let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. Denote

$$\Delta f_i = f(x_i) - f(x_{i-1}) \quad (i = 1, \dots, n).$$

If there exists a positive number M such that

$$\sup \sum_{i=1}^{n} \left|\Delta f_i\right| \leq M,$$

where the supremum is taken among all partitions of $[a, b]$, then f is said to be of bounded variation on $[a, b]$, or briefly speaking, bounded variation function (BV function).

Now, some results immediately follow from the definition, as shown in the next two theorems.

Theorem 3. *If f is monotonic, then f is of bounded variation.*

Proof. For every partition of $[a, b]$ we have

$$\begin{cases} \Delta f_i \geq 0, \text{if } f \text{ is increasing} \\ \Delta f_i \leq 0, \text{if } f \text{ is decreasing} \end{cases}.$$

Hence, we get

$$\sum_{i=1}^{n} \left|\Delta f_i\right| = \sum_{i=1}^{n} \left|f(x_i) - f(x_{i-1})\right| = \left|\sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1})\right)\right| = \left|f(b) - f(a)\right|.$$

This completes the proof. □

Theorem 4. *If f is continuous on $[a, b]$ and if f' exists and is bounded in (a, b), then f is of bounded variation.*

Proof. Since f' is bounded in (a, b), there is a positive number A such that $\left|f'(x)\right| \leq A$. For every partition of $[a, b]$, it follows by Mean Value Theorem that there exist $t_i \in (x_{i-1}, x_i)$, such that $f(x_i) - f(x_{i-1}) = f'(t_i)(x_i - x_{i-1})$. This gives

$$\left|\Delta f_i\right| \leq A(x_i - x_{i-1}),$$

and

$$\sum_{i=1}^{n} \left|\Delta f_i\right| \leq A(b - a),$$

which implies f is of bounded variation. □

In both cases discussed above, we have shown some sufficient conditions for functions to be of bounded variation if the functions feature the requirements. Next theorem demonstrates what necessary condition needs to be satisfied if a function is of bounded variation.

Theorem 5. *If f is a bounded variation function, then f is bounded.*

Proof. Since f is of bounded variation, there is a positive number M such that $\sum |\Delta f_i| \leq M$. To show f is bounded, we hope there is a positive number A such that $|f(x)| \leq A$ holds for every $x \in [a,b]$. Now, consider partition $P = \{a,x,b\}$(where $a < x \leq b$), the hypothesis give us

$$|f(x) - f(a)| + |f(b) - f(x)| \leq M.$$

This implies $|f(x) - f(a)| \leq M$, and it follows that

$$|f(x)| \leq |f(a)| + M. \tag{1}$$

Note that (1) also holds for $x = a$, thus it holds for all $x \in [a,b]$. □

After being acquainted with some bounded variation functions, we shall proceed to other more important topics and find characterizations of bounded variation functions. However, we have to introduce "total variation" of a bounded variation function in order to know more about this kind of functions.

Definition 6 (total variation)**.** Let f be a bounded variation function on $[a,b]$, and let P be a partition of $[a,b]$. We write

$$S(f;P) = \sum_{i=1}^{n} |\Delta f_i|.$$

The number

$$V_f(a,b) = \sup\left\{S(f;P) : P \text{ is a partition of } [a,b]\right\}$$

is called the total variation of f on the interval $[a,b]$. Sometimes, the notation will be shortened to V_f when there is no ambiguity.

It is worth to note that V_f must be finite, since f is of bounded variation. In the following discussions, we are going to study some properties of total variation as a function of f, in other words, we study how does $V_f(a,b)$ behave as f varies.

Theorem 7. *Let f,g be functions of bounded variation. Then so are their sum, difference, and product. Moreover, we have*

$$V_{f\pm g} \leq V_f + V_g \quad \textit{and} \quad V_{f\cdot g} \leq \|g\|_{\sup} \cdot V_f + \|f\|_{\sup} \cdot V_g,$$

where

$$\|\phi\|_{\sup} = \sup_{x\in[a,b]} |\phi(x)|,$$

for $\phi = f,g$.

Proof. Let a partition P of $[a,b]$ be given. Note that

$$\begin{aligned}&\sum_{i=1}^{n}\left|(f(x_i)\pm g(x_i))-(f(x_{i-1})\pm g(x_{i-1}))\right|\\ \le&\sum_{i=1}^{n}\left|f(x_i)-f(x_{i-1})\right|+\sum_{i=1}^{n}\left|g(x_i)-g(x_{i-1})\right|\\ \le&\, V_f+V_g.\end{aligned}$$

Hence, V_f+V_g is an upper bound of $\sum\left|(f(x_i)\pm g(x_i))-(f(x_{i-1})\pm g(x_{i-1}))\right|$. This implies $f\pm g$ are of bounded variation and that $V_{f\pm g}\le V_f+V_g$. Now, let $h(x)=f(x)\cdot g(x)$. Then, we have

$$\begin{aligned}|\Delta h_i|&=|f(x_i)g(x_i)-f(x_{i-1})g(x_{i-1})|\\ &\le|f(x_i)g(x_i)-f(x_{i-1})g(x_i)|+|f(x_{i-1})g(x_i)-f(x_{i-1})g(x_{i-1})|\\ &\le\|g\|_{\sup}\cdot|\Delta f_i|+\|f\|_{\sup}\cdot|\Delta g_i|.\end{aligned}$$

We conclude that

$$\sum_{i=1}^{n}|\Delta h_i|\le\sum_{i=1}^{n}\left(\|g\|_{\sup}\cdot|\Delta f_i|+\|f\|_{\sup}\cdot|\Delta g_i|\right)\le\|g\|_{\sup}\cdot V_f+\|f\|_{\sup}\cdot V_g.$$

This gives that $\|g\|_{\sup}\cdot V_f+\|f\|_{\sup}\cdot V_g$ is an upper bound of $\sum|\Delta h_i|$ for all partitions. Therefore, $f\cdot g$ is of bounded variation and

$$V_{f\cdot g}\le\|g\|_{\sup}\cdot V_f+\|f\|_{\sup}\cdot V_g.$$

The proof is completed. □

Remark. Theorem 7 shows that the set V of all functions of bounded variation on $[a,b]$ is a linear space. In fact, Theorem 11 (which we will discuss later) indicates that $V\subseteq S$ if S is any linear space which contains all monotonic functions on $[a,b]$.

Now, we wonder whether f/g is of bounded variation provided that both f and g are of bounded variation. However, if we do further observation on f/g, it is easy to see that f/g might not even bounded. Yet, if we assume that g is bounded away from zero, which literally means that the values of g would not be arbitrarily close to 0, then f/g is of bounded variation. Now, we write down this observation in a more precise and mathematical way.

Theorem 8. *Let f and g are of bounded variation. We assume that g is bounded away from zero, that is, there exists a positive number m such that $0<m\le|g(x)|$ for all $x\in[a,b]$. Then, f/g is of bounded variation. Moreover, we have*

$$V_{f/g}\le\frac{V_f}{m}+\frac{\|f\|_{\sup}V_g}{m^2}.$$

Proof. Let a partition P of $[a,b]$ be given and let $h(x) = f(x)/g(x)$.

$$\begin{aligned}
|\Delta h_i| &= \left|\frac{f(x_i)}{g(x_i)} - \frac{f(x_{i-1})}{g(x_{i-1})}\right| = \left|\frac{f(x_i)g(x_{i-1}) - g(x_i)f(x_{i-1})}{g(x_i)g(x_{i-1})}\right| \\
&\le \frac{\big(|f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_{i-1})| + |f(x_{i-1})g(x_{i-1}) - g(x_i)f(x_{i-1})|\big)}{|g(x_i)g(x_{i-1})|} \\
&\le \frac{|\Delta f_i|}{|g(x_i)|} + \frac{\|f\|_{\sup} |\Delta g_i|}{m^2} \le \frac{|\Delta f_i|}{m} + \frac{\|f\|_{\sup} |\Delta g_i|}{m^2}.
\end{aligned}$$

This implies

$$\sum_{i=1}^{k} |\Delta h_i| \le \sum_{i=1}^{k} \left(\frac{|\Delta f_i|}{m} + \frac{\|f\|_{\sup} |\Delta g_i|}{m^2} \right) \le \frac{V_f}{m} + \frac{\|f\|_{\sup} V_g}{m^2}.$$

This completes the proof. □

Next, we are going to study the properties of total variation $V_f(a,x)$ as a function of x. Before we start the discussion, we shall prove a theorem which is so called additive property of total variation.

Theorem 9 (additive property of total variation)**.** *Let f be of bounded varaition on $[a,b]$, and assume that $c \in (a,b)$. Then f is of bounded variation on $[a,c]$ and on $[c,b]$. Moreover, we have*

$$V_f(a,b) = V_f(a,c) + V_f(c,b).$$

Proof. Let P_1 and P_2 be partition of $[a,c]$ and $[c,b]$, respectively. Note that $P_0 = P_1 \cup P_2$ is a partition of $[a,b]$. We have

$$S(f;P_1) + S(f;P_2) = S(f;P_0) \le V_f(a,b). \tag{2}$$

Now, if we fix the partition P_2, then we have

$$V_f(a,c) \le V_f(a,b) - S(f;P_2)$$

by taking supremum on the left hand side. Taking supremum again on the left hand side of the inequality

$$S(f;P_2) \le V_f(a,b) - V_f(a,c)$$

gives the conclusion that

$$V_f(a,c) + V_f(c,b) \le V_f(a,b).$$

Note that for every partition P of $[a,b]$, we have $S(f;P) \le S(f;P')$, where $P' = P \cup \{c\}$. Let $P_1 = P' \cap [a,c]$ and let $P_2 = P' \cap [c,b]$. Then,

$$S(f;P) \le S(f;P') = S(f;P_1) + S(f;P_2) \le V_f(a,c) + V_f(c,b).$$

Thus, $V_f(a,c) + V_f(c,b)$ is an upper bound of $\{S(f;P) : P \text{ is a partition of } [a,b]\}$. We conclude that $V_f(a,c) + V_f(c,b) \geq V_f(a,b)$. This completes the proof. $\square$

Theorem 10. *Let f be a function of bounded variation on $[a,b]$, and let $V(x)$ denotes the function $V_f(a,x)$. ($V_f(a,a)$ is defined to be 0.) Then, both V and $V - f$ are monotonically increasing.*

Proof. If $a \leq x < y \leq b$, then Theorem 9 gives $V_f(a,x) + V_f(x,y) = V_f(a,y)$. This implies $V(y) - V(x) = V_f(x,y) \geq 0$, and therefore V is increasing. To prove $V - f$ is increasing, let $D(x) = V(x) - f(x)$ on $[a,b]$. If $a \leq x < y \leq b$, then

$$D(y) - D(x) = (V(y) - V(x)) - (f(y) - f(x)) = V_f(x,y) - (f(y) - f(x)).$$

But it follows from the definition of total variation that $V_f(x,y) \geq f(y) - f(x)$. Thus, we conclude that $D(y) \geq D(x)$ and D is increasing. $\square$

Theorem 10 suggests what sufficient and necessary conditions need to be met for a function to be a bounded variation function.

Theorem 11. *Let f be a function defined on $[a,b]$. Then, f is of bounded variation if and only if f can be expressed as the difference of two increasing functions.*

Proof. If f is of bounded variation, then both V and $D = V - f$ are increasing (from Theorem 10). We have $f = V - D$ are the difference of two increasing function. Conversely, if f can be expressed as the difference of two increasing functions, then it follows from Theorem 3 and Theorem 7 that f is of bounded variation. $\square$

3 Problems

Here are some problems provided in Apostol, T.M.'s book, most of which attracted my interest.

Problem 1. *A function f, defined on $[a,b]$, is said to satisfy a uniform Lipschitz condition of order $\alpha > 0$ on $[a,b]$ if there exists a constant $M > 0$ such that $|f(x) - f(y)| < M|x - y|^\alpha$ for all x and y in $[a,b]$.*

1. *If f is such a function, show that $\alpha > 1$ implies f is constant on $[a,b]$, whereas $\alpha = 1$ implies f is of bounded variation.*

2. *Give an example of a function f satisfying a uniform Lipschitz condition of order $\alpha < 1$ on $[a,b]$ such that f is not of bounded variation.*

3. *Give an example of a function g which is of bounded variation on $[a,b]$ but which satisfies no uniform Lipschitz condition on $[a,b]$.*

Solution 1.

1. Suppose f is a non-constant function satisfies a uniform Lipschitz condition of order $\alpha > 1$ on $[a, b]$. Since f is non-constant, there are two numbers $p < q$ in $[a, b]$ such that $f(p) \neq f(q)$. Let s and t denote the number $|p - q|$ and $|f(p) - f(q)|$, respectively. We consider a finite sequence $\{x_k\}_{k=0}^n$ explicitly defined by $x_k = p + ks/n$. The triangle inequality and the hypotheses of a uniform Lipschitz condition give us

$$\begin{aligned}|f(p) - f(q)| \leq \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq \sum_{k=1}^{n} M\, |x_k - x_{k-1}|^\alpha \\ \implies t \leq n \cdot M \cdot \left(\frac{s}{n}\right)^\alpha .\end{aligned}$$

   The last inequality fails to be true whenever $n > (M/t)^{\frac{1}{\alpha-1}} \cdot s^{\frac{\alpha}{\alpha-1}}$. Hence, we conclude that a function f satisfies a uniform Lipschitz condition of order $\alpha > 1$ must be constant.

   Now, let g be a function satisfies a uniform Lipschitz condition of order 1, and let an arbitrary partition P be given. Note that

$$\sum_{i=1}^{n} |\Delta g_i| \leq \sum_{i=1}^{n} M^* (x_i - x_{i-1}) = M^*(b - a),$$

   which indicates g is of bounded variation. (M^* is used for the purpose of distinguishing itself from M.) Another approach see the subproblems *1* and *2* in Problem 2.

2. To solve this problem, we shall first prove some lemmas.

   Lemma 1. If f is a function defined on two closed intervals I, J with following properties:

   - The intersection of I and J contains at most 1 point (which implies that the intersection point must be an endpoint of these intervals, if the intersection point exist).
   - $f(I) \subset f(J)$.
   - f satisfies a uniform Lipschitz condition of order α on both I and J.

   Then, f satisfies a uniform Lipschitz condition of order α on $I \cup J$.

   Proof. Since f satisfies a uniform Lipschitz condition of order α on both I and J, there exist M_1, M_2 such that

$$|f(x) - f(y)| \le M_1|x-y|^\alpha, \text{ for all } x, y \in I.$$
$$|f(x) - f(y)| \le M_2|x-y|^\alpha, \text{ for all } x, y \in J.$$

Now, we claim that $|f(x) - f(y)| \le M|x-y|^\alpha$, for all $x, y \in I \cup J$, where $M = \max\{M_1, M_2\}$. It is easy to see that we only need to prove that $|f(x)-f(y)| \le M|x-y|^\alpha$ for all $x \in I$ and $y \in J$. From our hypothesis that $f(I) \subset f(J)$, there exists $x' \in J$ such that $f(x) = f(x')$. Note that

$$|f(x) - f(y)| = |f(x') - f(y)| \le M_2|x'-y|^\alpha \le M_2|x-y|^\alpha,$$

the last inequality holds from our first assumption. This proves the lemma. □

Lemma 2. If x, y are two distinct numbers in $[0,1]$, and if $\alpha \in (0,1)$, then $|x^\alpha - y^\alpha| \le |x-y|^\alpha$.

This implies that $f(x) = x^\alpha$ satisfies a uniform Lipschitz condition of order α.

Proof. Without loss of generality, we assume that $x > y$. For every $y \in [0,1)$, we consider the function

$$\phi_y(x) = (x-y)^\alpha - x^\alpha + y^\alpha \qquad (x \ge y).$$

It is easy to see that $\phi_y(y) = 0$ and that

$$\frac{d\phi_y}{dx} = \alpha(x-y)^{\alpha-1} - \alpha x^{\alpha-1} \qquad (x > y).$$

Since $\alpha \in (0,1)$, we have $d\phi_y(x)/dx > 0$ whenever $x > y$. Note that $\phi_y(x)$ is continuous at y, hence $\phi_y(x)$ is strictly increasing, we conclude that $\phi_y(x) > 0$ for $x > y$. This proves the lemma. □

Lemma 3. Given $\epsilon > 0$. There exists a function f defined on $[a,b]$, such that:

- f satisfies a uniform Lipschitz condition of order α $(0 < \alpha < 1)$.
- $1 \le V_f(a,b) < \infty$ and $f(a) = f(b) = 0$.
- There exists a positive number $\delta < \epsilon$ such that $f([a,b]) = [0,\delta]$.

Proof. Let $l = b-a$ and let n be a positive integer such that

$$(2n)^{1-\alpha} \cdot l^\alpha \ge 1 \quad \text{and} \quad \left(\frac{l}{2n}\right)^\alpha < \epsilon.$$

Now, we divide $[a,b]$ into $2n$ intervals

$$I_k = \left[a + \frac{(k-1)l}{2n}, a + \frac{kl}{2n}\right] \quad (k = 1, 2, \dots, 2n).$$

We define f as follows:

$$f(x) = \begin{cases} \left(x - a - \dfrac{(m-1)l}{n}\right)^{\alpha} & \text{, if } x \in I_{2m-1} \\ \left(a + \dfrac{ml}{n} - x\right)^{\alpha} & \text{, if } x \in I_{2m} \end{cases} \qquad (m = 1, 2, \dots, n).$$

It is easy to see that f is increasing on every interval I_{2m-1} and it is decreasing on every interval I_{2m}. Moreover, we have $f(I_k) = [0, (l/2n)^{\alpha}]$ (this verify the third requirement of the function f) and f satisfies a uniform Lipschitz condition of order α on each interval I_k. It follows from Lemma 1 and Lemma 2 that f satisfies a uniform Lipschitz condition of order α on $[a, b]$. (In fact, we have $|f(x) - f(y)| \leq |x - y|^{\alpha}$ from the proof of Lemma 1.) This completes the proof. □

After proving these lemmas, we can begin to prove the main problem. We construct a function f defined on $[0, 1]$, satisfying following properties:

(a) $f(0) = 0$ and $f(1/n) = 0$, for all $n \in \mathbb{N}$.

(b) $|f(x) - f(y)| \leq |x - y|^{\alpha}$, for all $x, y \in [1/(n+1), 1/n]$.

(c) $V_f(1/(n+1), 1/n) > 1$, for all $n \in \mathbb{N}$.

(d) $f([1/(n+2), 1/(n+1)]) \subset f([1/(n+1), 1/n])$.

This is possible by Lemma 3. We claim that f satisfies a uniform Lipschitz condition of order α on $[0, 1]$ but is not of bounded variation. If f is of bounded variation, then it follows from Theorem 9 that

$$V_f(0, 1) = V_f\left(0, \frac{1}{n}\right) + V_f\left(\frac{1}{n}, 1\right) > n - 1 \quad (n \in \mathbb{N}).$$

It contradicts the assumption that $V_f(0, 1)$ is bounded. From the construction of f and Lemma 1, we have

$$|f(x) - f(y)| < |x - y|^{\alpha}, \text{ for all } x, y \in (0, 1].$$

We also have $|f(x)| \leq x^{\alpha}$. (It is easy to verify the construction in Lemma 3.) We conclude that f satisfies a uniform Lipschitz condition of order α on $[0, 1]$. This shows f meets the requirements for the problem.

3. Define $g : [0, e^{-1}] \to [0, 1]$ as:

$$g(x) = \begin{cases} \sqrt{\dfrac{-1}{\ln x}} & \text{, if } x \in (0, e^{-1}] \\ 0 & \text{, if } x = 0 \end{cases}.$$

Now, note that the function g is continuous and monotonic, we conclude that g is of bounded variation from Theorem 3. Suppose g satisfies a uniform Lipschitz condition on $[0, e^{-1}]$ of order $0 < \alpha \leq 1$. Then, there exists $M > 0$ such that $|g(x) - g(y)| < M|x - y|^{\alpha}$ for all $x, y \in [0, e^{-1}]$. In particular,

$$\sqrt{\frac{-1}{\ln x}} < Mx^{\alpha} \tag{3}$$

holds for all $x \in (0, e^{-1}]$. This inequality is equivalent to

$$M^2 x^{2\alpha} \ln\left(\frac{1}{x}\right) > 1.$$

The substitution $x = e^{-t}$ turns (3) into

$$\frac{M^2 t}{e^{2\alpha t}} > 1, \; t \in [1, \infty). \tag{4}$$

However, it follows from L'Hospital's Rule that $\lim_{t\to\infty}(M^2t)/e^{2\alpha t} = 0$, which contradicts (4). Thus, g satisfies no uniform Lipschitz condition on $[0, e^{-1}]$ and therefore g meets the requirements of the problem.

Remark. The inverse of the function constructed in the subproblem *3* in Problem 1 is $g^{-1} : [0,1] \to [0, e^{-1}]$ defined by

$$g^{-1}(x) = \exp\left(\frac{-1}{x^2}\right), \; x \in (0,1], \text{ and } \quad g^{-1}(0) = 0.$$

This function $h = g^{-1}$ has some quite interesting properties:

1. The nth derivative of h at 0 exists. Moreover, $h^{(n)}(0) = 0$.
2. The Maclaurin series of h does not converge to h, although it converges everywhere on $\mathbb{R}$.

Problem 2. *A function f, defined on $[a, b]$, is said to be absolutely continuous, if for every $\epsilon > 0$ there is a $\delta > 0$ such that*

$$\sum_{k=1}^{n} \left| f(b_k) - f(a_k) \right| < \epsilon,$$

for every n ($n \in \mathbb{N}$) disjoint open subintervals (a_k, b_k) of $[a, b]$, the sum of whose length

$$\sum_{k=1}^{n} (b_k - a_k) < \delta.$$

Prove the following statements.

1. *Every absolutely continuous function on $[a, b]$ is continuous and of bounded variation.*
2. *If f satisfies a uniform Lipschitz condition of order 1, then f is absolutely continuous.*

Solution 2.

1. Let f be a function which is absolutely continuous on $[a,b]$. Given $\epsilon > 0$, then there exists a $\delta > 0$ satisfying the condition described above. If $x < y$ are two points in $[a,b]$ such that $y - x < \delta$, then $|f(x) - f(y)| < \epsilon$ (from the definition of absolutely continuous). This implies f is uniformly continuous and thus continuous.

   Now, we shall show that f is of bounded variation. Let a partition P of $[a,b]$ be given. Let N be a positive integer such that $b - a < N\delta$. Consider a refinement P' of P, where P' is defined as:
$$P' = P \cup \left\{ a + \frac{(b-a)}{N}, a + \frac{2(b-a)}{N}, \dots, a + \frac{(N-1)(b-a)}{N} \right\}.$$
   Let
$$P_k = P' \cap \left[a + \frac{(k-1)(b-a)}{N}, a + \frac{k(b-a)}{N} \right] \quad (k = 1, 2, \dots, N).$$
   It is easy to see that $S(f;P_k) < \epsilon$ for each $k = 1, 2, \dots, N$ (from the definition of absolutely continuous). We conclude that
$$S(f;P') = \sum_{k=1}^{N} S(f;P_k) < N\epsilon.$$
   Since P' is a refinement of P, we have $S(f,P) \le S(f;P') < N\epsilon$, this shows f is of bounded variation.

2. Since f satisfies a uniform Lipschitz condition of order 1, there exists a constant $M > 0$ such that $|f(x) - f(y)| < M|x - y|$ for all x, y in $[a,b]$. Given $\epsilon > 0$. Choose $\delta = \epsilon/M$. Let (a_k, b_k) $(k = 1, 2, \dots, n)$ be n $(n \in \mathbb{N})$ disjoint open subintervals of $[a,b]$, the sum of whose length
$$\sum_{k=1}^{n} (b_k - a_k) < \delta.$$
   Then, by some simple estimations, we get
$$\sum_{k=1}^{n} \left| f(b_k) - f(a_k) \right| \le \sum_{k=1}^{n} M(b_k - a_k) < M\delta = \epsilon.$$
   We conclude that f is absolutely continuous on $[a,b]$.

Remark. Although it seems that a function being of bounded variation and continuous would be more likely to be absolutely continuous. However, there exists a function which is of bounded variation and continuous but not absolutely continuous. Cantor function is an example of these kind of functions. [3]

Problem 3. *Show that a polynomial f is of bounded variation on every compact interval $[a, b]$. Describe a method for finding the total variation of f on $[a, b]$ if the zeros of the derivative f' are known.*

Solution 3. We first prove the case when $f_n(x) = x^n$ for all $n \in \mathbb{N}$. f_1 is monotonic on $[a, b]$ and thus is of bounded variation (by Theorem 3). Suppose the statement is true for all $n \leq k$, where k is a positive number. Then, $f_{k+1}(x) = f_k(x) \cdot f_1(x)$, hence f_{k+1} is of bounded variation (from Theorem 7). By induction, each $f_n(x) = x^n$ ($n \in \mathbb{N}$) is of bounded variation. It is easy to see that f is of bounded variation implies that cf is of bounded variation, where c is a real constant. (We could replace M to $|c| \cdot M$ in Definition 2 to show cf is of bounded variation.) Also, note that a constant function is of bounded variation. Now, for each polynomial f with degree n, we have

$$f(x) = \sum_{i=0}^{n} c_i x^i,$$

and therefore it is sum of finitely many functions of bounded variation. This implies that f is of bounded variation. (Theorem 7.)

If f' is zero, then f is constant and $V_f(a, b) = 0$. Suppose f' is not 0, and let $Z_{f'}$ be the set $\{t \in (a, b) : f'(t) = 0\}$. Since f' is a polynomial, $Z_{f'}$ is a finite set. Let X denote the set $Z_{f'} \cup \{a, b\}$. Because X is finite and contains at least 2 elements, we write $X = \{a = t_0 < t_1 < \cdots < t_m = b\}$. By Theorem 9, we have

$$V_f(a, b) = \sum_{i=1}^{m} V_f(t_{i-1}, t_i).$$

Now, note that the image under f' of (t_{i-1}, t_i) cannot contain both positive and negative numbers, otherwise the continuity of f' would indicate that there exists a number $s \in (t_{i-1}, t_i)$ such that $f'(s) = 0$ (Intermediate Value Theorem), which contradicts $s \notin Z_{f'}$. Hence, either $f'(x) > 0, \forall x \in (t_{i-1}, t_i)$ or $f'(x) < 0, \forall x \in (t_{i-1}, t_i)$ holds, thus f is monotonic on (t_{i-1}, t_i). We conclude that $V_f(t_{i-1}, t_i) = |f(t_{i-1}) - f(t_i)|$, therefore

$$V_f(a, b) = \sum_{i=1}^{m} |f(t_{i-1}) - f(t_i)|.$$

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